The Sunity Representation to Improve the Accuracy of Some Computations

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The Sunity Representation to Improve the Accuracy of Some Computations

Problem: some computations using the floating-point representation produce a significant **loss of accuracy**

Solution: **a new representation**, which can be used together with the floating-point representation

Outline

- Motivation: Relative error amplification in FP
- Starting point: *The unity representation for unit range numbers*
  - Asilomar 2005
- Symmetric Unity (*Sunity*) Representation
- Some examples
- Drawbacks and advantages
- Conclusions

Motivation

- **Floating-point representation**: Large dynamic range
- Relative representation error (maximum at each binade):
  - constant over the whole normalized range
  - *rel error = 0.5 ulp*
- In some computations:
  - Amplification of the relative error with respect to the argument errors
  - Loss of accuracy

Relative Error Increase in FP

- For instance, these classes of computations can produce relative error increase:
  - Cancellation: x-y

$$R_{x-y} = (x R_x + y R_y) \times \frac{1}{|x-y|} \approx x(R_x + R_y) \times \frac{1}{|x-y|}$$

    - In case of cancellation: x-y small, $R_{x\text{-}y}$ large
  - Functions with large $x \; f'(x) / f(x)$

$$R_{f(x)} = | f'(x) \frac{x}{f(x)} | R_x$$

- Relative error of arguments is *amplified*

Relative Error Increase in FP. Some examples

- Single precision FP
- Cancellation x-y, with x→y

$x = 2^{-2}+2^{-3}+2^{-5}+2^{-22}+2^{-27}$ $\qquad R_x = 1.23 \times 2^{-26}$

$y = 2^{-2}+2^{-3}+2^{-5}+2^{-26}$ $\qquad R_y = 1.23 \times 2^{-25}$

$z = FP(x-y) = 2^{-23}+2^{-24}+2^{-25}$ $\qquad R_z = 1.54 \times 2^{-4}$

- Function with large $x\, f'(x)/f(x)$:
  - $\boldsymbol{f(x) = x^n, \quad R_{f(x)} = n\, R_x}$

$x = 2^{-2}+2^{-3}+2^{-5}+2^{-22}+2^{-27}$ $\qquad R_x = 1.23 \times 2^{-26}$

$z = FP(x^{70}) = 2^{-91}+2^{-97}+2^{-99}+2^{-100}+2^{-104}$ $\qquad R_z = 1.35 \times 2^{-20}$

How to avoid the loss of accuracy?

- Algorithm level
  - Change the algorithms
  - Appropriate library functions

  - But …
    - Very specific approaches
    - Require awareness of each case
    - Particular analysis needed

- Implementation level
  - More precision on the arguments
  - Double, quad precision

  - But …
    - Expensive in terms of hardware resources and computation time
    - Not sufficient in some cases

How to avoid the loss of accuracy?

- Our approach: Hardware solution to reduce the loss of accuracy
  - New data representation
  - Modification of functional units
  - Programmer intervention not necessary
- Applicable to arguments and results around the value 1
  - Important computations in this range: trigonometric, exponential, ….

Special case: Arguments or Results close to 1

- Example: 1-cos(x), x→0

$$x = 1.00000000000000000000000 \times 2^{-5}$$

$$\cos(x) = 1.11111111110000000000000 \times 2^{-1} \qquad \textbf{\textit{(SP FP)}}$$

$$1-\cos(x) = 1.11111111111100000000000 \times 2^{-12} \qquad \textbf{\textit{(SP FP)}}$$

$$1-\cos(x) = 1.1111111111110\color{blue}{1010101010} \times 2^{-12} \qquad \textbf{\textit{(Maple)}}$$

Special case: Arguments or Results close to 1

- Example: arccos(x), $x \rightarrow 1$, $x = \cos(\theta)$ (SP FP)

$$\theta = 1.00000000000000000000000 \times 2^{-5}$$

$$\cos(\theta) = 1.11111111110000000000001 \times 2^{-1}$$

$$\arccos(\cos(\theta)) = 1.11111111111111{\color{blue}010101011} \times 2^{-6}$$

- Example: ln(x), $x \rightarrow 1$

$$x = 1 + 2^{-23} + 2^{-26}$$

$$\ln(x) = 1.11111111111111111111111 \times 2^{-24} \quad \textbf{\textit{(SP FP)}}$$

$$\ln(x) = 1.000{\color{blue}11111111111111111111} \times 2^{-23} \quad \textbf{\textit{(Maple)}}$$

Computations Producing Relative Error Amplification

- Cancellations $1-f(x)$ with $f(x) \to 1$
  - $1-x, \quad x \to 1$
  - $1-\cos(x), \quad x \to 0$
  - $e^x-1, \quad x \to 0$
- Without cancellation
  - $\arccos(x), \quad x \to 1$
  - $\ln(x), \quad x \to 1$
  - $(1+x)^n, \quad x \to 0$

Our Approach

- Representation with very low relative error close to 1
  - Use this representation to avoid large relative errors in some computations
- Provide hardware support
  - Applicable to wider class of applications
  - Automatic use
  - Programmer control also possible
  - Can be used in general purpose processors, DSPs, GPUs, …

Our Approach

- Limitations and disadvantages
  - Modification of operations and library functions
  - Added complexity for the operations, and for using operands and results
    - Could affect the performance
    - Not beneficial in some cases
    - Option: *Disable* the representation
  - One bit to indicate the representation being used
    - Reduce precision or exponent range

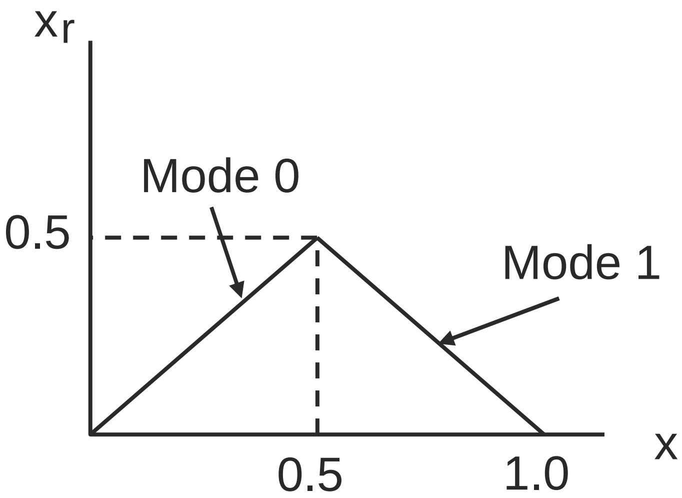
Starting Point: the Unity Representation

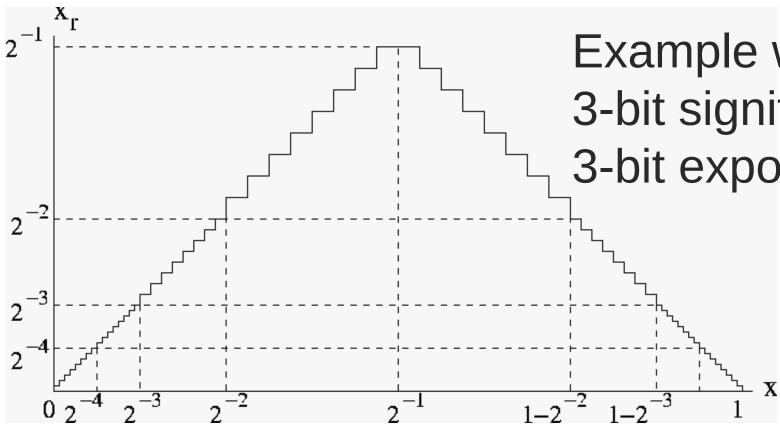
- Improved accuracy for numbers 1.0-ε
- Better accuracy for some functions using unit-range variables
- Functions that benefit from the Unity repr.:
  - Trigonometric
  - Exponential
  - Calculation of 3D rotation angle
  - ….
- *Asilomar 2005*

Unity Representation

- Symmetric density around 0.5
  - x real number in [0,1),
  - x_r unity representation of x (x_r is a FP number)

$$x_r = \begin{cases} fp\ round(x) & if\ 0 \le x < 0.5 \quad (\text{mode } 0) \\ fp\ round(1-x) & if\ 0.5 \le x < 1 \quad (\text{mode } 1) \end{cases}$$





Example with 3-bit significand, 3-bit exponent

Unity Representation

- Example: $1.0 – (2^{-23} + 2^{-27} + 2^{-35}+2^{-40})$

  FP: $1.11111111111111111111110 \times 2^{-1}$

  Unity: $1-1.00010000000100001000000 \times 2^{-23}$

- Example: $1.0 - (2^{-63} + 2^{-67} + 2^{-75}+2^{-80})$

  FP: $1.00000000000000000000000$

  Unity: $1-1.00010000000100001000000 \times 2^{-63}$

An Example: Arc cosine using Standard FP and Unity

- Example: arccos(cos(angle))
  - Representations simulated with Maple
  - fp format results calculated using 40 digits
- Standard FP:
  - Angle: $\theta = 1.000100000000000100001000 \times 2^{-15}$
  - Round. cosine: $\cos(\theta) = 1.000000000000000000000000$
  - Result: $\arccos(\cos(\theta)) = 0$
- Unity:
  - Angle: $\theta = 1.000100000000000100001000 \times 2^{-15}$
  - Round. cosine: $\cos(\theta) = 1 - \boxed{1.001000010000001000110001 \times 2^{-31}}$
  - Result: $\arccos(\cos(\theta)) = 1.000100000000000100001000 \times 2^{-15}$

Unity representation of the cosine

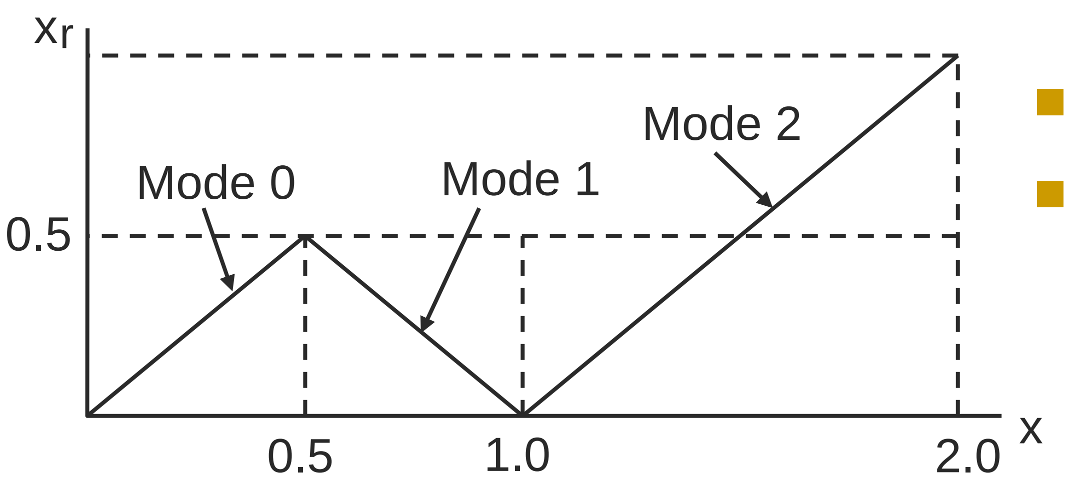
Sunity Representation vs. Unity Representation

- Limitations of the Unity representation
  - Requires special instructions
  - Does not allow values in other ranges
  - Does not allow large accuracy for values close to but larger than 1
- Sunity representation:
  - Extension of the unity representation
  - Includes values larger than 1
  - Wider application
  - Integrated with the FP representation

Symmetric Unity (*Sunity*) Representation

- Symmetric density around 1.0
  - x real number in [0,2),
  - x_r unity representation of x

$$x_r = \begin{cases} fp\ round(x) & if\ 0 \le x < 0.5 & (\text{mode } 0) \\ fp\ round(1-x) & if\ 0.5 \le x < 1 & (\text{mode } 1) \\ fp\ round(x-1) & if\ 1 \le x < 2 & (\text{mode } 2) \end{cases}$$



- X_r, FP number
- In [0.5,2) represents the displacement from 1
  - In [1,2), positive displacement

Symmetric Unity (*Sunity*) Representation

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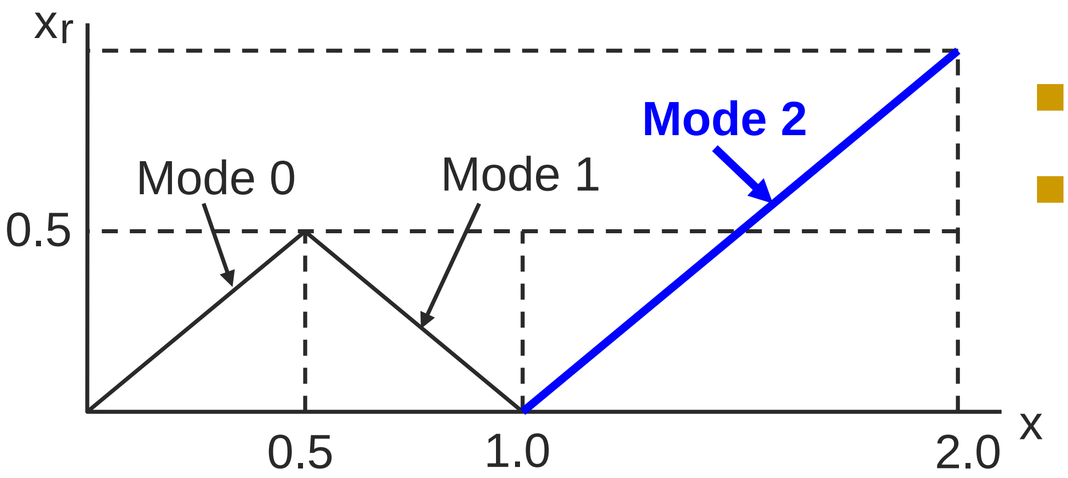

- X_r, FP number
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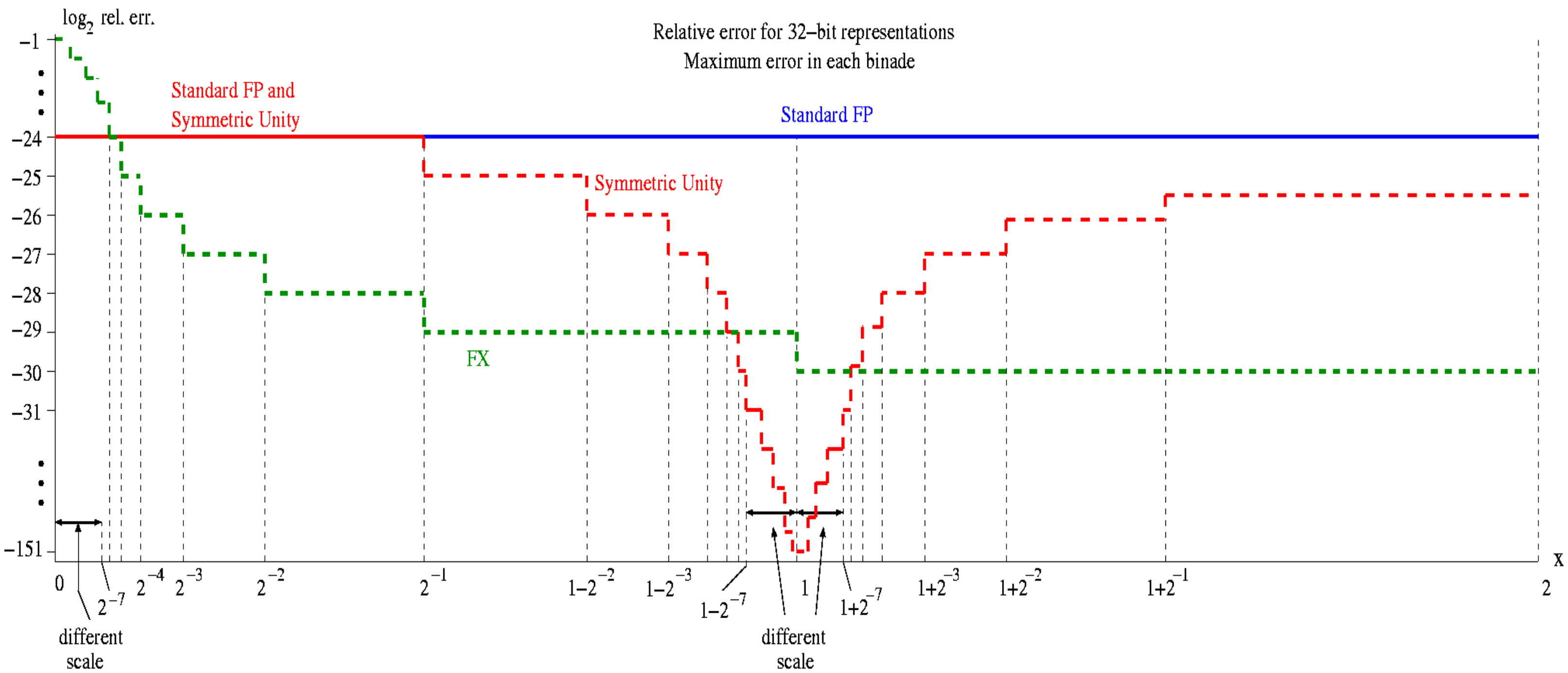
  - In [1,2), positive displacement

Relative Error Reduction with the Sunity Representation



- Significant reduction of relative error around 1
- For $x = 1 + a \times 2^{-d}, \, with\, 1 \le |a| < 2, \, 2^{-u} : ulp$
  - FP: $R_x \le \min(0.5 \times 2^{-u}, a \times 2^{-d})$
  - Sunity: $R_x \le 2^{-(d+u)} \; (for \;\, x = 0.5)$

Relative Error Reduction with the Sunity Representation

- For $x = 1 + a \times 2^{-d}, \ with \ 1 \le |a| < 2$
  - FP:

2^{-u}

1: 1.0000 … 0000

$a \times 2^{-d}$: 1.xxxx … xxxx

1.xxxx…xxxx

$R_x = 0.5 \times 2^{-u}$

$R_x = a \times 2^{-d}$

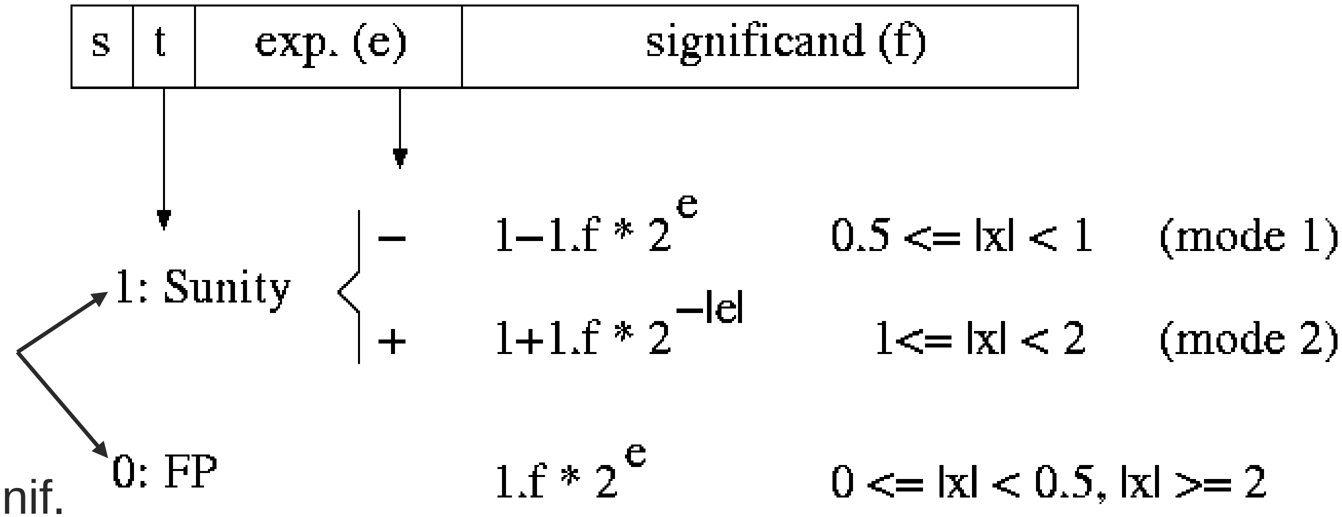
Then $R_x \le \min(0.5 \times 2^{-u}, a \times 2^{-d})$

  - Sunity:

$$x_r = round(a \times 2^{-d})$$

Then $R_x \le 0.5 \times 2^{-(d+u)} / x_{\min} = 2^{-(d+u)} \ (for \ x = 0.5)$

Combining Sunity and FP Representations



Loose one bit with respect to standard FP.
It can be taken out from the exponent or signif.

- Improved accuracy around 1
- No special instructions
- Instruction uses operands and produces results in sunity representation
- FP and sunity in the same instruction

Computations that Benefit from the Sunity Representation

Type of computation	Computation	Relative error FP	Relative error Sunity
	x y	$min(0.5 \times 2^{-u}, a \times 2^{-d})$ 0.5×2^{-u}	$2^{-d} \times 2^{-u}$ 0.5×2^{-u}
Type 1	$1 - x$ $1 - cos(y)$ $e^y - 1$	$min(0.5 \times 2^d \times 2^{-u}, 1)$ $min(0.5 \times 2^{2d+1} \times 2^{-u}, 1)$ $min(0.5 \times 2^d \times 2^{-u}, 1)$	2^{-u} 2^{-u} 2^{-u}
Type 2	$\pi/2 - arccos(y)$ $= (\pi/2)(1 - [(2/\pi)arccos(y)])^*$	$2^{d+1} \times 2^{-u}$	1.5×2^{-u}
Type 3	$arccos(x)$ $ln(x)$	$min(0.25 \times 2^d \times 2^{-u}, 1)$ $min(0.5 \times 2^d \times 2^{-u}, 1)$	0.5×2^{-u} 2^{-u}
Type 4	$(1 + y)^n$	$0.5 \times 2^{-u} +$ $n \times min(0.5 \times 2^{-u}, y)$	$0.5 \times 2^{-u} +$ $n \times 2^{-d} \times 2^{-u}$
Type 5	$(b + y)^n$, b exact	$0.5 \times 2^{-u} + (n/b) \times$ $(0.5 \times 2^{-u} + min(0.5 \times 2^{-u}, y))$	$2^{-u} + n \times 2^{-d} \times 2^{-u}$

* Term between [ ] is sunity

x: value close to 1, $x = 1 + a \times 2^{-d}$

y: value close to 0, $y = a \times 2^{-d}$

Computations that Benefit from the Sunity Representation

- Example: *z=x-1, with* $x = 1 + a \times 2^{-d}$

$$R_z = \frac{R_x \times x}{|x-1|} \approx \frac{R_x}{a \times 2^{-d}}$$

- FP:

$$R_z = \frac{R_x(fp)}{a \times 2^{-d}} = \min(0.5 \times 2^{d} \times 2^{-u}, 1)$$

- Sunity:

$$R_z = \frac{R_x(su)}{a \times 2^{-d}} = 2^{-u}$$

Computations that Benefit from the Sunity Representation

- Example: *z=1-cos(y), with* $y = a \times 2^{-d}$

$$R_z = \frac{R_{\cos(y)} \times \cos(y)}{1 - \cos(y)} \qquad \cos(y) = \frac{1 - y^2}{2} = 1 - a^2 \times 2^{-(2d+1)}$$

Then $$R_z = \frac{R_{\cos(y)}}{a^2 \times 2^{-(2d+1)}}$$

- FP: $R_z = \min(0.5 \times 2^{2d+1} \times 2^{-u}, 1)$
- Sunity: $R_z = 2^{-u}$

Example of Transformation to Sunity

- *z=π/2-arccos(y), with* $y = a \times 2^{-d}$
  - Cancellation for y close to 0
- FP:

$$R_z = \frac{(\pi/2) \times R_{\pi/2} - \arccos(y) \times R_{\arccos(y)}}{\left|\pi/2 - \arccos(y)\right|}$$

Then $R_z = 2^{(d+1)} \times 2^{-u}$

- Transformation to sunity: $z = (\pi/2) \times (1 - [(2/\pi)\arccos(y)])$

Then $$R_z = R_{\pi/2} + \frac{R_{(2/\pi)\arccos(y)}}{a \times 2^{-d}} = 1.5 \times 2^{-u}$$

- This result can be obtained in FP if function *π/2-arccos(y)* is available

More Examples: Rotation Angle

- R: 3 x3 rotation matrix
- θ=arccos((trace(R)-1)/2)
  - trace(R)=a+b+c, with a,b,c<=1
  - a,b,c close to 1.0 ➔ $\cos(\theta) \approx 1$ (small θ)
  - standard FP: reduced accuracy

- Example: a, b, c = 1 - ($2^{-20} + 2^{-29} + 2^{-39}$)

  Maple result with 40 decimal digits rounded to fp single:

  $$\theta = 1.10111011110101101001100 \times 2^{-10}$$

  In FP: $\cos(\theta) = 1.11111111111111111100101 \times 2^{-1}$

  $$\theta = 1.1\boxed{1010110010011010101011} \times 2^{-10}$$

Rotation Angle with Sunity Representation

- Operand representation (for a,b,c >= 0.5) :

a_r=1-a, b_r=1-b, c_r=1-c

- Algorithm:

cos(θ) = (a+b)/2 + (c-1)/2

unity = unity + unity

y + z

For (a+b)/2 >= 0.5, y_r=1- (a+b)/2 =(a_r+b_r)/2 (mode 1)

For c >= 0.5, z_r=- c_r/2 (mode 0)

For cos(θ) >= 0.5,

$\cos(\theta)_r = 1 - (y+z) = 1 - (1-y_r + z_r) = y_r - z_r$ (mode 1)

arccos(cos(θ)) special function with sunity operand

Rotation Angle with Sunity Representation

$a, b, c = 1 - (2^{-20} + 2^{-29} + 2^{-39}) >= 0.5$

$a_r, b_r, c_r = 2^{-20} + 2^{-29} + 2^{-39}$ (mode 1)

$y_r = [(a+b)/2]_r = (a_r + b_r)/2 = 2^{-20} + 2^{-29} + 2^{-39}$ (mode 1)

$z_r = [(1-c)/2]_r = -c_r/2 = -(2^{-21} + 2^{-30} + 2^{-40})$ (mode 0)

$\cos(\theta)_r = 1 - (y+z) = y_r - z_r$

$= 2^{-20} + 2^{-21} + 2^{-29} + 2^{-30} + 2^{-39} + 2^{-40}$ (mode 1)

$\arccos(\cos(\theta)) = 1.10111011110101101001100 \times 2^{-10}$

(special function with sunity operand)

(same result as maple with 40 digits rounded to fp single)

Limitations and disadvantages

- Modification of operations and library functions
- Added complexity for the operations, and for using operands and results
  - Not beneficial in some cases
  - Could affect the performance
  - Option: *Disable* the representation
- One bit to indicate the representation being used
  - Reduce precision or exponent

Conclusions

- Presented representation with very low relative error close to 1
- Showed improved accuracy for some computations
- Uses one bit to distinguish between representations
  - from exponent or significand
- More complex implementation of operations and functions

Future Work

- Apply to more complex computations with combined fp and sunity variables
- Perform implementation of operations and functions
- Extend the representation to other ranges