Decidability of Collision between a Helical Motion and an Algebraic Motion

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Joint work with:

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Zero Problem

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- We need decidability with TM!

Algebraic Problems

- $\alpha \in \mathbb{C}$ is *algebraic*, if $p(\alpha) = 0$ for some nontrivial $p \in \mathbb{Z}[x]$.
  - natural numbers, rational numbers, $\sqrt{2}$, i, $\ldots$
  - finitely representable $\longrightarrow$ *countable*
  - closed under $\pm$, $\times$, $\div$, $\texttt{RootOf}()$

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- Decidable (qualitatively) *if* a zero problem for *algebraic* expression is decidable. (e.g. relative configuration of line & circle)
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- *Decidable in TM-sense.*

Constructive Root Bound

- Classical bound: $\alpha = \sqrt{3} - \sqrt{2}$, then Cauchy's bound says $|\alpha| \geq \frac{1}{11}$ if $\alpha \neq 0$ (α is a zero of $x^4 - 10x^2 + 1$)
- How to use:
  - Suppose we have: $|\alpha| \geq B$ if $\alpha \neq 0$.
  - Compute a numerical approximation $\tilde{\alpha}$ of α so that $|\tilde{\alpha} - \alpha| < B/2$. (# bits to be calculated is $\log_2(B/2)$.)
  - If $|\tilde{\alpha}| \geq B$, then $\mathrm{sign}\,(\alpha) = \mathrm{sign}\,(\tilde{\alpha})$. Otherwise, $\alpha = 0$.
- Some modern bounds: Degree-Measure [Mignotte (1982)], Degree-Height & Degree-Length [Yap-Dubé (1994)], BFMS [Burnikel et al (1989)], Eigenvalue [Scheinerman (2000)], Conjugate [Li-Yap (2001)], BFMSS [Burnikel et al (2001)], k-ary [Pion-Yap (2002)]
- $\alpha = \sqrt{x} + \sqrt{y} - \sqrt{x + y + 2\sqrt{xy}}$, $\quad x = a/b$, $y = c/d$, $\quad a, b, c, d$: L-bit integers
  $\rightarrow$ The number of bits sufficient to determine the zero problem for α: 28L + 60 (Li-Yap), 96L + 30 (BFMSS), . . .
- *No general bound for transcendental expressions!*

Exact Geometric Computation (EGC)

- The most successful approach to nonrobustness.
- Exact determination of discete or *geometric* relations. (e.g. Is a point on a line?, Does a plane cut a sphere? convex hull, Voronoi diagram, . . .)
- Philosophy: algorithm = sequence of steps, step = either construction or test, test = determines the branching path, combinatorial relations are determined by path $\Rightarrow$ If all comparisons are correct, then we take the correct path (exact geometric relation). $\longrightarrow$ Constructive root bound is at its heart!
- Exploits numerical approximation. $\longrightarrow$ fast and adaptive
- Implementations: LEDA, CGAL, Core (You can use standard algorithms.)
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Terminology:
decidable = (Turing) computable = decidable in EGC sense = $\cdots$

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- Only a few examples which is TM decidable.

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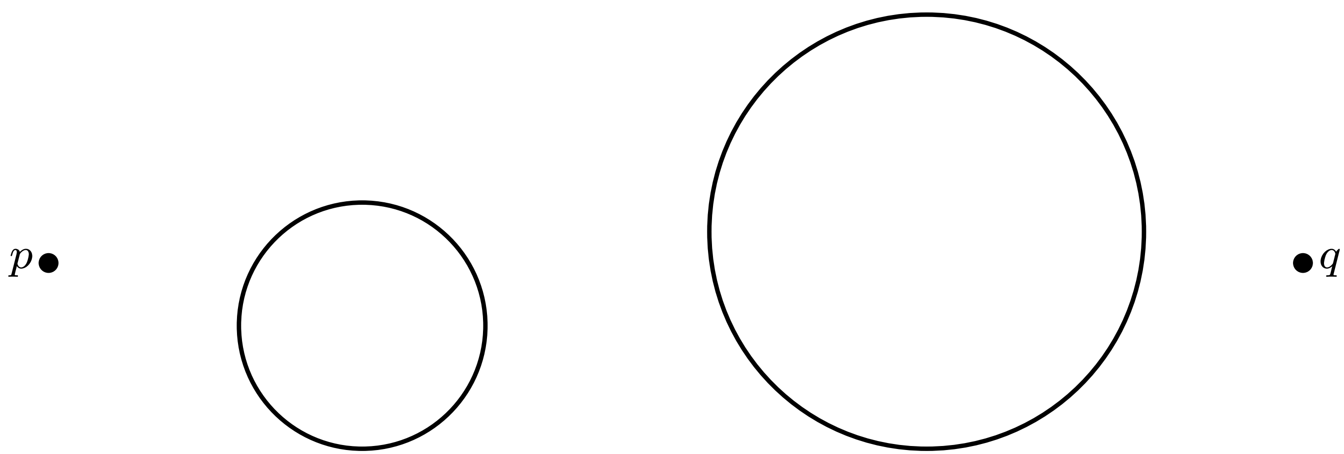

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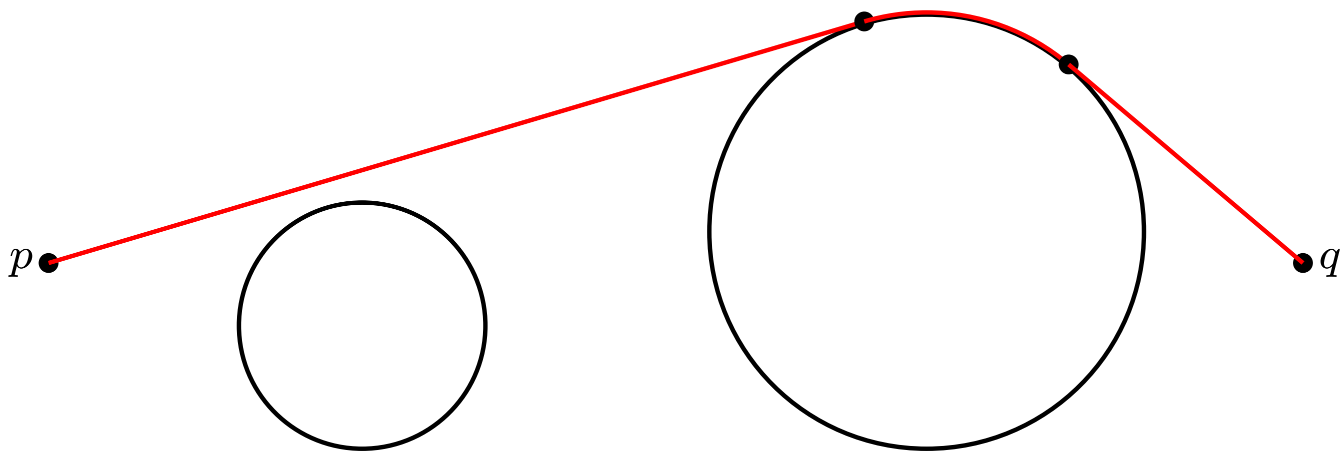

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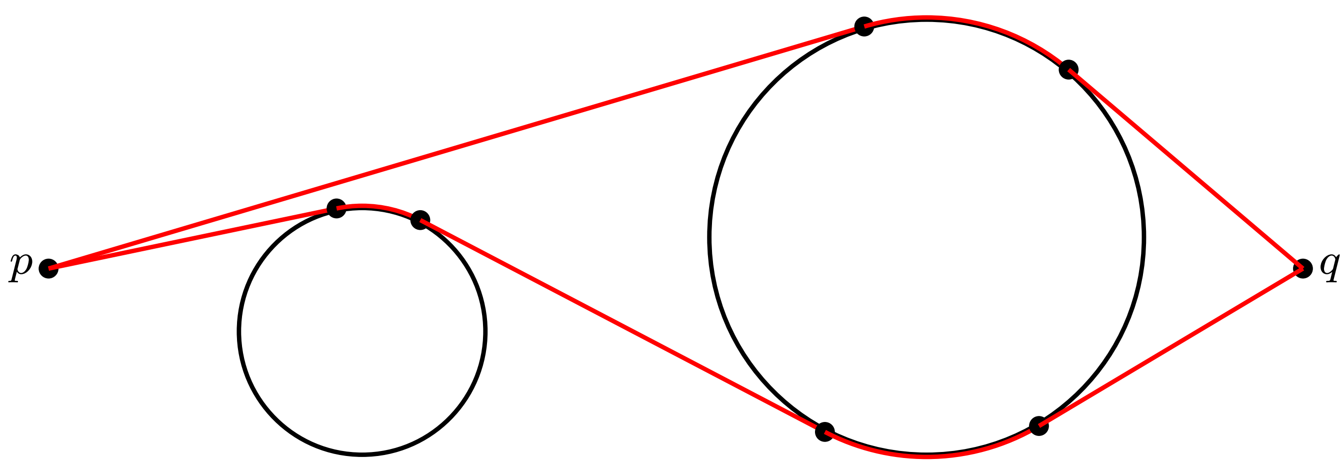

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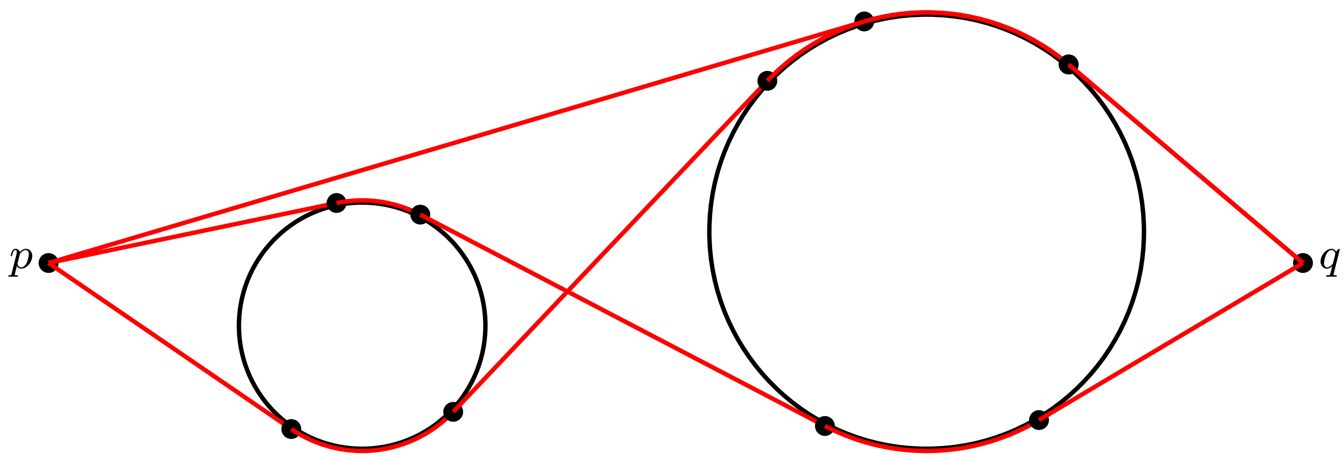

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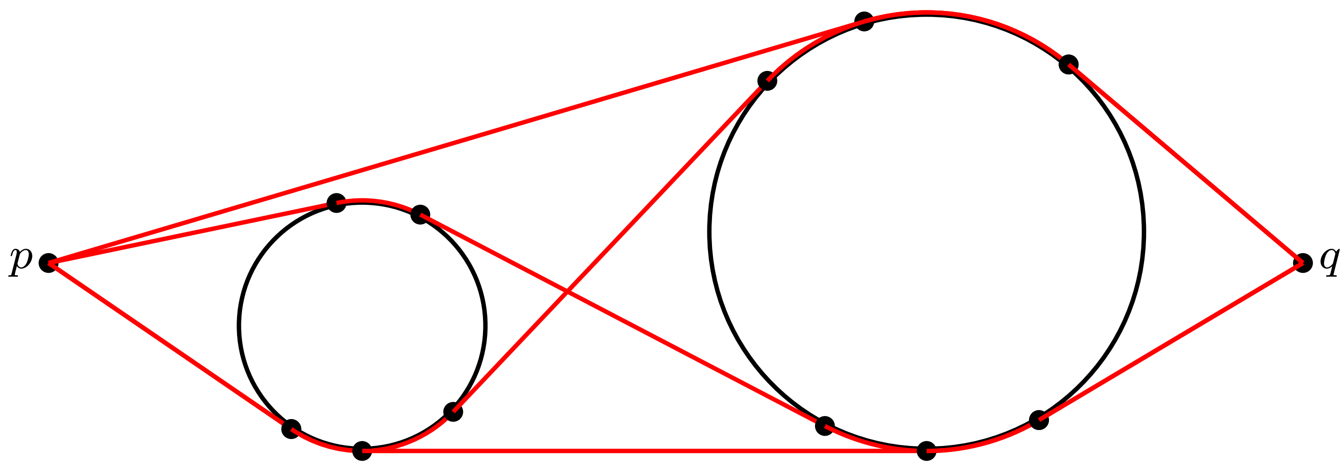

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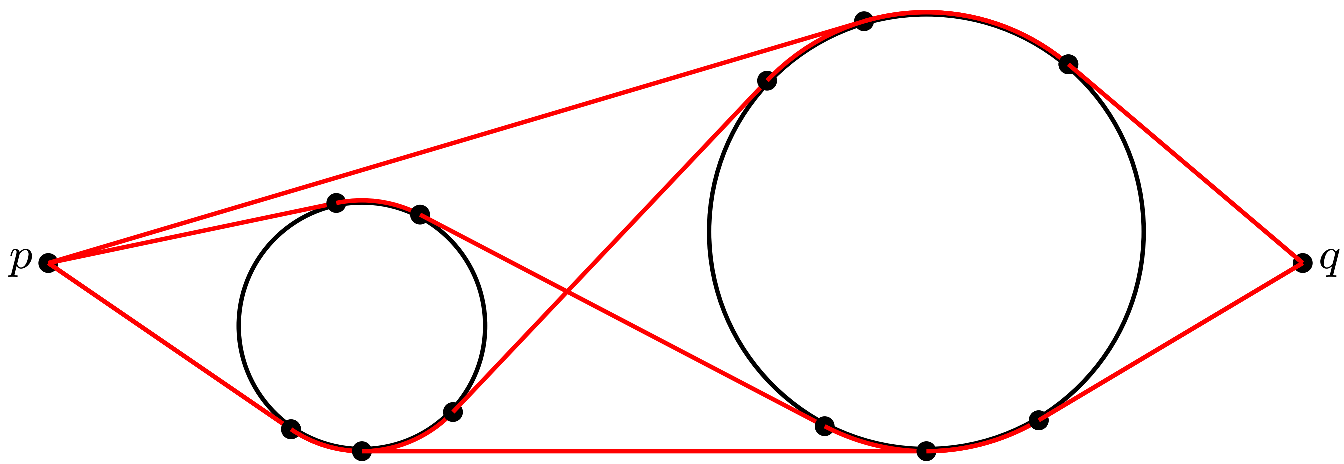

- Assume: each coord. of p, q, centers of C_i, radii of C_i are all algebraic.
- Seemingly a typical problem in computational geometry – *feasible paths*.
- *The first* nontrivial example of a transcendental problem which turned out to be TM decidable. [Chang et al, to appear in IJCGA]

Length of Feasible Path

- Find Feasible Paths: $\mu = \mu_1; \mu_2; \cdots; \mu_k$
  - Alternating between line segments and circular arcs
  - Boundary points are *algebraic*.
  - Sum up the lengths of $\begin{cases} \text{line segments:} & \sqrt{(\cdot - \cdot)^2 + (\cdot - \cdot)^2} \\ \text{circular arcs:} & r \cdot \theta \end{cases}$

$$\rightarrow \quad d(\mu) = \sum_i d(\mu_i) = \sum_j \alpha_j + \sum_k r_k \theta_k$$

  - $\sum \alpha_j$: length of line segments $\Rightarrow$ algebraic
  - $\sum r_k \theta_k$: length of circular arcs
  - $\cos \theta_k$: algebraic $\Rightarrow$ θ_k: transcendental (Lindemann's Lemma)

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Comparison of Two Feasible Paths:

$$d(\mu_1) - d(\mu_2) \rightarrow \alpha + r_1\theta_1 + \cdots + r_n\theta_n \quad \alpha, r_i\text{: algebraic}, \theta_i\text{: transcendental}$$

Decidability

We have to solve the zero problem for:

$$\begin{aligned}\overline{\Lambda} &= \alpha + r_1\theta_1 + \cdots + r_n\theta_n \\ &= \boxed{\alpha + (\pm i r_1)\log\left(\cos\theta_1 \pm i\sqrt{1-\cos^2\theta_1}\right) + \cdots + (\pm i r_1)\log\left(\cos\theta_1 \pm i\sqrt{1-\cos^2\theta_1}\right)}\end{aligned}$$

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Baker's Theorem Let $\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ be nonzero algebraic numbers, with their degrees $\leq d$ and heights $\leq H$. let

$$\Lambda = \alpha_0 + \alpha_1 \log\beta_1 + \cdots \alpha_n \log\beta_n \quad \text{(linear forms in logarithms).}$$

If $\Lambda \neq 0$, then $\exists$ constant $C = C(n, d, H)$ s.t. $|\Lambda| > 2^{-C}$.

Consequence: Λ is transcendental if $\Lambda \neq 0$.

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- So the problem is *transcendental* but *decidable*!
- How many bits are needed to solve the zero problem?

Effective Bound from Transcendental Number Theory

Theorem. (Waldschmidt) For $n \geq 2$, let $\gamma_0, \gamma_1, \cdots, \gamma_n$ be algebraic numbers, and let $\beta_1, \cdots, \beta_n$ be nonzero algebraic numbers. If

$$\Lambda := \gamma_0 + \gamma_1 \log \beta_1 + \cdots + \gamma_n \log \beta_n \neq 0,$$

then

$$|\Lambda| > \exp\{-2^{8n+51} n^{2n} D^{n+2} V_1 \cdots V_n (W + \log(EDV_n^+))(\log(EDV_{n-1}^+))(\log E)^{-n-1}\},$$

where

$$\begin{aligned} D &\geq [\mathbb{Q}(\gamma_0, \gamma_1, \cdots, \gamma_n, \beta_1, \cdots, \beta_n) : \mathbb{Q}], & W &\geq \max_{0 \leq j \leq n} \{\mathrm{ht}(\gamma_j)\}, \\ V_j &\geq \max\{\mathrm{ht}(\beta_j), |\log \beta_j|/D, 1/D\}, & V_1 &\leq \cdots \leq V_n, \\ V_{n-1}^+ &= \max\{V_{n-1}, 1\}, & V_n^+ &= \max\{V_n, 1\}. \\ 1 < E &\leq \min\{e^{DV_1}, \min_{1 \leq j \leq n} \{4DV_j/|\log \beta_j|\}\}. \end{aligned}$$

Bit Complexity

Some Definitions. $\alpha \in \mathbb{C}$: algebraic & $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$: its *minimal polynomial*

- Degree: $\deg(\alpha) := \deg(p) = n$
- Absolute logarithmic height: $h(\alpha) := \frac{1}{\deg(\alpha)} \log M(\alpha)$
- Mahler measure: $M(\alpha) := |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}$, where $\alpha_1, \cdots, \alpha_n$ are all the conjugates of α.

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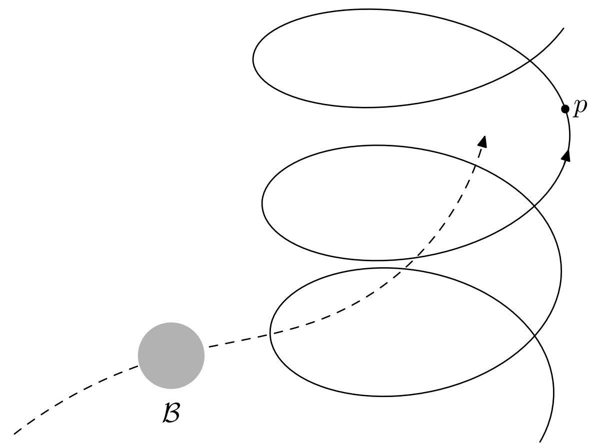
Bit Complexity:

- Assume the input is L*-bit rational numbers* (P/Q, where P, Q are L-bit integers. ($|P|, |Q| < 2^L$)), and N is the number of discs.
- Detailed estimation gives: $|\overline{\Lambda}| > \exp\left[-2^{O(N^2 + N \log L)}\right]$.
- The number of bits we need to expand to compare the lengths of two feasible paths is $2^{O(N^2 + N \log L)}$.

Our Problem

Given a helical motion $h(t) = (\cos t, \sin t, s \cdot t)$ of a point p and an algebraic motion $c(t) = (c_1(t), c_2(t), c_3(t))$ of a ball $\mathcal{B}$ with radius r, determine whether they will collide.



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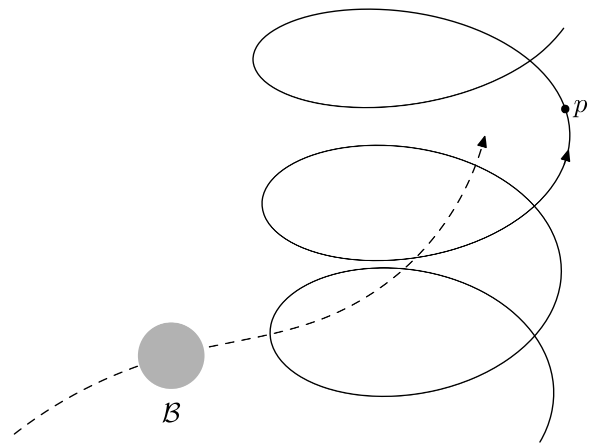

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  - $c_i(t)$ algebraic, if $\exists P(x, y) \in \mathbb{Z}[x, y]\ s.t.\ P(c_i(t), t) \equiv 0$
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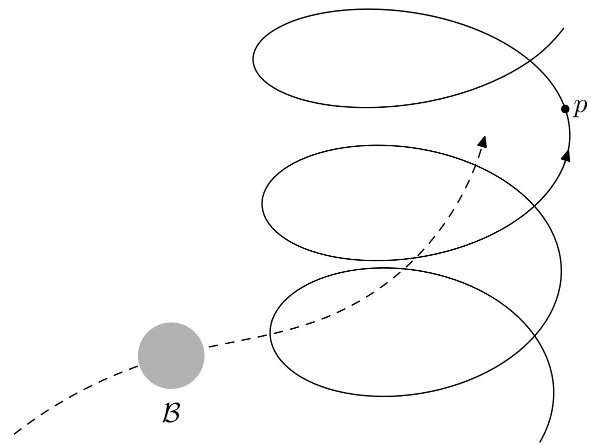

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- Natural question (e.g. in CAD)
- If both motions are algebraic $\longrightarrow$ becomes an algebraic problem.
- Turns out to be another (the second) nontrivial *transcendental* problem which is *decidable with TM*.

How?

$$?\exists t,\ ||h(t)-c(t)|| \leq r$$

Natural assumption: no collision initially

$$\begin{aligned}\Leftrightarrow ?\exists t,\ r^2 &= ||h(t)-c(t)||^2 \\ &= -2c_1(t)\cos t - 2c_2(t)\sin t + \left\{c_1(t)^2 + c_2(t)^2 - c_3(t)^2 + s\cdot t + 1\right\}\end{aligned}$$

$$\Leftrightarrow ?\exists t,\ a(t)\cos t + b(t)\sin t + d(t) = 0$$

$$\Leftrightarrow \begin{cases} ?\exists t,\ a(t) = b(t) = d(t) \quad \rightarrow \text{ algebraic problem} \\ ?\exists t,\ \frac{a(t)}{\sqrt{a(t)^2+b(t)^2}}\cos t + \frac{b(t)}{\sqrt{a(t)^2+b(t)^2}}\sin t = -\frac{d(t)}{a(t)^2+b(t)^2} \end{cases}$$

$$\Leftrightarrow ?\exists t,\ \cos\left(t \pm \arccos(\alpha(t))\right) = \delta(t)$$

$$\Leftrightarrow ?\exists t,\ t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) = 0 \mod 2\pi$$

$$\boxed{\Leftrightarrow ?\exists t,\ t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) + 2k\pi = 0, \quad (k\text{: between zeros of } \delta(t) \pm 1)}$$

Zero Problem Again

$$F(t) := t \pm \arccos(\alpha(t)) \pm \arccos(\delta(t)) + 2k\pi$$

$\longrightarrow$ Determine (exactly) the signs of all extremal points of F.

An extremal point t_* satisfy:

$$\begin{aligned} F'(t_*) &= 1 \pm \frac{\alpha'(t_*)}{\sqrt{1-\alpha(t_*)^2}} \pm \frac{\delta'(t_*)}{\sqrt{1-\delta(t_*)^2}} = 0 \\ \text{or} \quad \alpha(t_*) \pm 1 &= 0 \\ \text{or} \quad \delta(t_*) \pm 1 &= 0 \end{aligned}$$

$\longrightarrow$ t_* is algebraic.

$\longrightarrow$ Determine the sign of:

$$\begin{aligned} F(t_*) &= t_* \pm \arccos(\alpha(t_*)) \pm \arccos(\delta(t_*)) + 2k \arccos(-1) \\ &= \boxed{t_* \pm i \log\left\{\alpha(t_*) \pm i\sqrt{1-\alpha(t_*)}\right\} \pm i \log\left\{\delta(t_*) \pm i\sqrt{1-\delta(t_*)}\right\} \pm 2ki \log(-1)} \end{aligned}$$

$\longrightarrow$ Linear forms in logarithms! $\longrightarrow$ Decidable by Baker's Theorem

Bit Complexity

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  - all are L-bit rational numbers.
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- By Waldscmidt's theorem, we get:
  - $|F(t_*)| > \exp\left[-O\left(L^3 \log L \cdot N^2 8(\log N)^{13}\right)\right]$, if $F(t_*) \neq 0$.
  - We need $O\left(L^3 \log L \cdot N^2 8(\log N)^{13}\right)$ bits to solve the zero problem for one $F(t_*)$. $\rightarrow$ polynomial time!

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Merci! **Thanks!**