Software techniques for perfect elementary functions in floating-point interval arithmetic

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Outline

Interval arithmetic, directed rounding and elementary functions

Perfect interval functions

Implementation

Conclusion and future work
Introduction:
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Directed rounding modes

In this talk, we target **double-precision** computations.

- The IEEE-754 standard defines, for a real number \( x \) and a floating-point format \( \mathbb{F} \),
  - the correct rounding up of \( x \),
  - the correct rounding down of \( x \)
  - the correct rounding to zero of \( x \)

- Correct directed rounding supported by most processors at a cost for \( \pm, -, \times, \div, \sqrt{\cdot} \)

- Intended mostly for **interval arithmetic** (IA)
  - IA represents a real number \( x \) by an interval containing \( x \).
  - Interval endpoints are **machine-representable** numbers, here in double-precision.

- Some interval arithmetic packages don’t use directed rounding at all
Interval arithmetic does not require correct rounding

Important properties of interval operations are, *in this order*:

- **Safety**
  - For all real values in the input interval(s), the real result belongs to the output interval (*containment property*).

- **Sharpness** (of the output interval)
  - Finite number representation $\implies$ output interval usually larger than the image of the input interval
  - Define the *tight* result as the smallest interval ensuring the containment property, considering the number format
  - Related to correct rounding in directed rounding modes

- **Performance**

<table>
<thead>
<tr>
<th>Perfect interval elementary function:</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Safe</td>
</tr>
<tr>
<td>- Tight</td>
</tr>
<tr>
<td>- Efficient</td>
</tr>
</tbody>
</table>
Interval people have long been writing interval elementary functions which are
- safe (usually),
- sharp (but not tight),
- and sometimes efficient

In the CRLibm project, we have been writing correctly rounded elementary functions in directed rounding modes, wondering
- Why? Who cares
- What should the interface be?

We should be writing perfect interval elementary functions instead.
Some IA packages

- Sun Microsystems’ SunStudio
  - Very efficient integrated language/compiler approach (Fortran95 and C++)
  - Sharp (almost tight) elementary functions
    ▶ paper by Priest in Arith 13
    ▶ tight for more than 99% of input intervals
  - No published proof (and not open-source).
  - Not portable.

- fi_lib then C-XST then filib++
  - Portable, does not use hardware directed rounding
  - Neither sharp nor efficient, even for basic operations
  - Elementary functions written from scratch and carefully proven (in German).
Some IA packages (2)

- INTLAB (a Matlab package)
  - A trick by Rump: use the (underspecified) \texttt{libm} to build safe interval functions
  - But inefficient, non-sharp, and still requires development and proof work

- Boost
  - Empty shell: C++ templates and policies
  - Generic: FP intervals, but also multiple-precision if needed
  - Portable
  - Sharpness and efficiency depending on how you fill the shell

- MPFI
  - Safe and tight (on top of MPFR)
  - Not optimised for double-precision
    - but then you get multiple-precision IA...
Perfect interval functions

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Implementing an interval function

Evaluate $f([x_-, x_+])$

- If the function $f$ is monotonic increasing
  - Round down $f(x_-)$
  - Round up $f(x_+)$
  - That’s all.

- If the function is monotonic decreasing
  - Left as an exercise for the reader

- If the function is not monotonic on the interval
  - Elementary functions are well-behaved:
    - Such cases are handled in the argument reduction.
  - It may actually save work
    - $\sin([-42, 17])$

In theory, nothing difficult, really.
Evaluate $y \approx f(x)$ to some precision $\varepsilon$

Is this information enough to ensure that rounding $y$ is equivalent to rounding $f(x)$?

- Sometimes yes,
- Sometimes no.
Rounding up a function

\[ \frac{y}{(1 \pm \varepsilon)} \]

Two options:
- Round up \( \frac{y}{(1 - \varepsilon)} \) (Krämer, Priest)
  - safe,
  - not tight (in the red case),
  - fast.
- Onion peeling strategy (CRLibm, idea by Ziv)
  1. *Rounding test* on \( y \) knowing \( \varepsilon \): Are we in the green case?
  2. If yes, return \( y \) rounded up.
  3. If no, compute \( y_2 \) with a smaller \( \varepsilon_2 \), and go back to 1.

- safe if termination is proven (elementary functions),
- tight,
- slower, but just a little
  - small cost of rounding test
  - restarting computation is expensive, but rare.
There are finitely many FP input numbers $x$.
Among them, finitely many are such that $f(x)$ is an FP number:
- 1 for exp, log, sine, cosine, tangent, ...
- 23 for log10,
- 2047 for log2,
- ...
Among the remaining ones, there is a minimum (worst-case) distance from $f(x)$ to an FP number.
Choosing $\bar{\varepsilon}$ smaller than this distance guarantees correct rounding.
Lefèvre and Muller have computed such “worst cases” for quite a few functions.
Why perfection is not very expensive

Both approaches for rounding up a function need

- An algorithm for evaluating $y$,
- with a proven error bound $\bar{\epsilon}$
  - That’s the difficult bit

Additional cost for a perfect interval function:

- Compute the TMD worst cases
  - (once for all)
- Design a rounding test
  - (already studied in projects such as MPFR and CRLibm)
- Write a second, more accurate computation (and prove its $\bar{\epsilon}_2$)
  - We have it in CRLibm, too
  - This second step is easier to write and prove than the first
  - Its average runtime cost is very low
TMD worst case still missing for some functions on some intervals.

- Statistical arguments gives us fair hope that our code always returns correctly rounded result.
- For IA, hope isn’t enough.

Never mind. Round up $y_2/(1 - \bar{e}_2)$ at the end of the second step (or, second rounding test)

- Statistics predict that it will probably never change the result.
- If it does, it will add one ulp.
- Not expensive in average, because in the second step.

For such functions with missing TMD worst cases, we get a probably perfect interval functions.

- The “probably” is on tightness, not on safety nor performance.
- More TMD work may prove tightness a posteriori.
Performance should be more or less in a factor two with respect to the scalar libm (Priest, Arith13)

- (more): context changes (argument passing, processor status)
- (less): work sharing (argument reduction, table prefetches, cache misses)
- (less): intrinsic parallelism, better pipeline usage
Implementation on an Itanium® 2-based platform

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Convert Itanium-optimised correctly rounded exp and log into a perfect interval function.
Parallel Estrin evaluation

Itanium has two parallel FMAs (*fused multiply and add*), and many many registers.

\[
\begin{align*}
  z2i &= zi * zi; \\
  p67i &= c6 + zi * c7; \\
  p45i &= c4 + zi * c5; \\
  p23i &= c2 + zi * c3; \\
  p01i &= \text{logirhi} + zi * c1; \\
  z4i &= z2i * z2i; \\
  p47i &= p45i + z2i * p67i; \\
  p03i &= p01i + z2i * p23i; \\
  p07i &= p03i + z4i * p47i; \\
  \text{logi} &= p07i + Ei * \log2h;
\end{align*}
\]

\[
\begin{align*}
  z2s &= zs * zs; \\
  p67s &= c6 + zs * c7; \\
  p45s &= c4 + zs * c5; \\
  p23s &= c2 + zs * c3; \\
  p01s &= \text{logirhs} + zs * c1; \\
  z4s &= z2s * z2s; \\
  p47s &= p45s + z2s * p67s; \\
  p03s &= p01s + z2s * p23s; \\
  p07s &= p03s + z4s * p47s; \\
  \text{logs} &= p07s + Es * \log2h;
\end{align*}
\]
Combined rounding test

Itanium allows to mix rounding modes and precisions at no cost.

```c
/* Tentatively set the result */
result.INF = ROUND_EXT_TO_DOUBLE_DOWN(logi);
result.SUP = ROUND_EXT_TO_DOUBLE_UP(logs);

/* Rounding test */
mantissai = GET_EXT_MANTISSA(logi);
mantissas = GET_EXT_MANTISSA(logs);
bitsi = mantissai & (0x7ff&(accuracymask));
bitss = mantissas & (0x7ff&(accuracymask));
infDone= (bitsi!=0) && (bitsi!=(0x7ff&(accuracymask)));
supDone= (bitss!=0) && (bitss!=(0x7ff&(accuracymask)));

/* Only one test, expected true */
if(__builtin_expect(infDone && supDone, TRUE))
    return result;

/* otherwise launch accurate computation */
```
Avoiding a squaring of the number of tests

In each function, a few tests for special cases
(subnormals, infinities, etc.)

- **Make common case fast:**
  - Fuse all the tests for both endpoints into one, with static prediction.
- If this branch is taken, simply call the existing CRLibm functions that round up and down.
### Compared Itanium 2 timings

<table>
<thead>
<tr>
<th>Library</th>
<th>exp</th>
<th>interval exp</th>
<th>log</th>
<th>interval log</th>
</tr>
</thead>
<tbody>
<tr>
<td>libm</td>
<td>42</td>
<td>n/a</td>
<td>31</td>
<td>n/a</td>
</tr>
<tr>
<td>fi_lib</td>
<td>586</td>
<td>1038</td>
<td>619</td>
<td>1158</td>
</tr>
<tr>
<td>CRlibm</td>
<td>60</td>
<td>69</td>
<td>66</td>
<td>96</td>
</tr>
</tbody>
</table>
An exp/log iteration

\[
\begin{cases}
  y_{n+1} = \log(x_n) \\
  x_{n+1} = \exp(y_{n+1})
\end{cases}
\]

<table>
<thead>
<tr>
<th>Library</th>
<th>Interval bloat in ulps</th>
</tr>
</thead>
<tbody>
<tr>
<td>fi.lib</td>
<td>13</td>
</tr>
<tr>
<td>CRlibm, RU and RD functions</td>
<td>2</td>
</tr>
<tr>
<td>CRlibm merged interval function</td>
<td>2</td>
</tr>
</tbody>
</table>
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Conclusions

- Itanium is nice for interval arithmetic
  - Nice floating-point unit for basic operations
  - Many many registers for parallel evaluation of elementary functions.
  - Predication, etc, for fused tests.
- CRLibm is nice for writing perfect interval functions
  - Safety: We are very happy to recycle our proofs
  - Tightness thanks to correct rounding
  - Performance-oriented algorithms
Future work

- Argue a lot: Is it worth writing tight functions?
  - Yes in CRLibm (why not? No additional work)
  - Not sure if you start from scratch

- A few implementation questions
  - The basic structure for FP intervals
  - Trigonometric argument reduction
  - Portability on a register-starved Pentium

- Publish CRLibm0.14beta1

Questions?