Software techniques for perfect elementary functions in floating-point interval arithmetic

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Outline

Interval arithmetic, directed rounding and elementary functions

Perfect interval functions

Implementation

Conclusion and future work

Introduction: Interval arithmetic, directed rounding and elementary functions

Directed rounding modes

In this talk, we target double-precision computations.

- The IEEE-754 standard defines, for a real number x and a floating-point format $\mathbb{F}$,
  - the correct rounding up of x,
  - the correct rounding down of x
  - the correct rounding to zero of x
- Correct directed rounding supported by most processors at a cost for $+, -, \times, \div, \sqrt{\ }$
- Intended mostly for interval arithmetic (IA)
  - IA represents a real number x by an interval containing x.
  - Interval endpoints are machine-representable numbers, here in double-precision.
- Some interval arithmetic packages don't use directed rounding at all

Interval arithmetic does not require correct rounding

Important properties of interval operations are, *in this order:*

- Safety
  - For all real values in the input interval(s), the real result belongs to the output interval (containment property).
- Sharpness (of the output interval)
  - Finite number representation $\Longrightarrow$ output interval usually larger than the image of the input interval
  - Define the tight result as the smallest interval ensuring the containment property, considering the number format
  - Related to correct rounding in directed rounding modes
- Performance

Perfect interval elementary function:

- Safe
- Tight
- Efficient

This article

- Interval people have long been writing interval elementary functions which are
  - safe (usually),
  - sharp (but not tight),
  - and sometimes efficient
- In the `CRLibm` project, we have been writing correctly rounded elementary functions in directed rounding modes, wondering
  - *Why? Who cares*
  - *What should the interface be?*
- We should be writing perfect interval elementary functions instead.

Some IA packages

- Sun Microsystems' SunStudio
  - Very efficient integrated language/compiler approach (Fortran95 and C++)
  - Sharp (almost tight) elementary functions
    - paper by Priest in Arith 13
    - tight for more than 99% of input intervals
  - No published proof (and not open-source).
  - Not portable.
- `fi_lib` then C-XST then `filib++`
  - Portable, does not use hardware directed rounding
  - Neither sharp nor efficient, even for basic operations
  - Elementary functions written from scratch and carefully proven (in German).

Some IA packages (2)

- INTLAB (a Matlab package)
  - A trick by Rump: use the (underspecified) `libm` to build safe interval functions
  - But inefficient, non-sharp, and still requires development and proof work
- Boost
  - Empty shell : C++ templates and policies
  - Generic: FP intervals, but also multiple-precision if needed
  - Portable
  - Sharpness and efficiency depending on how you fill the shell
- MPFI
  - Safe and tight (on top of MPFR)
  - Not optimised for double-precision
    - but then you get multiple-precision IA...

Perfect interval functions

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Implementing an interval function

Evaluate $f([x_-, x_+])$

- If the function f is monotonic increasing
  - Round down $f(x_-)$
  - Round up $f(x_+)$
  - That's all.
- If the function is monotonic decreasing
  - Left as an exercise for the reader
- If the function is not monotonic on the interval
  - Elementary functions are well-behaved:
    Such cases are handled in the argument reduction.
  - It may actually save work
    - $\sin([-42, 17])$

In theory, nothing difficult, really.

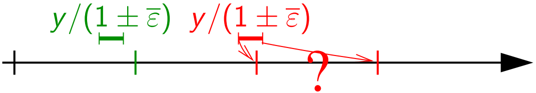
Rounding up a function

Evaluate $y \approx f(x)$ to some precision $\overline{\varepsilon}$

Is this information enough to ensure that rounding y is equivalent to rounding $f(x)$?

- Sometimes yes,
- Sometimes no.

Rounding up a function

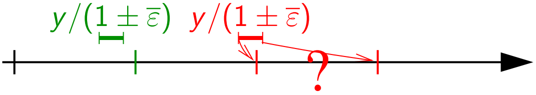


Two options:

- Round up $y/(1 - \overline{\varepsilon})$ (Krämer, Priest)
  - safe,
  - not tight (in the red case),
  - fast.
- Onion peeling strategy (`CRLibm`, idea by Ziv)
  1. *Rounding test* on y knowing $\overline{\varepsilon}$: Are we in the green case?
  2. If yes, return y rounded up.
  3. If no, compute y_2 with a smaller $\overline{\varepsilon}_2$, and go back to 1.

  - safe if termination is proven (elementary functions),
  - tight,
  - slower, but just a little
    - small cost of rounding test
    - restarting computation is expensive, but rare.

The Table Maker's Dilemma



- There are finitely many FP input numbers x
- Among them, finitely many are such that $f(x)$ is an FP number
  - 1 for exp, log, sine, cosine, tangent, ...
  - 23 for log10,
  - 2047 for log2,
  - ...
- Among the remaining ones, there is a minimum (worst-case) distance from $f(x)$ to an FP number.
- Choosing $\overline{\varepsilon}_2$ smaller than this distance guarantees correct rounding.
- Lefèvre and Muller have computed such "worst cases" for quite a few functions.

Why perfection is not very expensive

Both approaches for rounding up a function need

- An algorithm for evaluating y,
- with a proven error bound $\overline{\varepsilon}$
  - That's the difficult bit

Additional cost for a perfect interval function:

- Compute the TMD worst cases
  - (once for all)
- Design a rounding test
  - (already studied in projects such as MPFR and `CRLibm`)
- Write a second, more accurate computation (and prove its $\overline{\varepsilon}_2$)
  - We have it in `CRLibm`, too
  - This second step is easier to write and prove than the first
  - Its average runtime cost is very low

Bonus

- TMD worst case still missing for some functions on some intervals.
  - Statistical arguments gives us fair hope that our code always returns correctly rounded result.
  - For IA, hope isn't enough.
- Never mind. Round up $y_2/(1-\overline{\varepsilon}_2)$ at the end of the second step (or, second rounding test)
  - Statistics predict that it will probably never change the result.
  - If it does, it will add one ulp.
  - Not expensive in average, because in the second step.
- For such functions with missing TMD worst cases, we get a probably perfect interval functions.
  - The "probably" is on tightness, not on safety nor performance.
  - More TMD work may prove tightness *a posteriori*.

Performance of an interval function

Performance should be more or less in a factor two
with respect to the scalar libm (Priest, Arith13)

- (more): context changes (argument passing, processor status)
- (less): work sharing (argument reduction, table prefetches, cache misses)
- (less): intrinsic parallelism, better pipeline usage

Implementation on an Itanium® 2-based platform

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The present work

Convert Itanium-optimised correctly rounded `exp` and `log` into a perfect interval function.

Parallel Estrin evaluation

Itanium has two parallel FMAs *(fused multiply and add)*,
and many many registers.

```
z2i = zi*zi;                    z2s = zs*zs;
p67i = c6 + zi*c7;              p67s = c6 + zs*c7;
p45i = c4 + zi*c5;              p45s = c4 + zs*c5;
p23i = c2 + zi*c3;              p23s = c2 + zs*c3;
p01i = logirhi + zi*c1;         p01s = logirhs + zs*c1;
z4i  = z2i*z2i;                 z4s  = z2s*z2s;
p47i = p45i + z2i*p67i;         p47s = p45s + z2s*p67s;
p03i = p01i + z2i*p23i;         p03s = p01s + z2s*p23s;
p07i = p03i + z4i*p47i;         p07s = p03s + z4s*p47s;
logi = p07i + Ei*log2h;         logs = p07s + Es*log2h;
```

Combined rounding test

Itanium allows to mix rounding modes and precisions at no cost.

```
/* Tentatively set the result */
result.INF = ROUND_EXT_TO_DOUBLE_DOWN(logi);
result.SUP = ROUND_EXT_TO_DOUBLE_UP(logs);

/* Rounding test  */
mantissai = GET_EXT_MANTISSA(logi);
mantissas = GET_EXT_MANTISSA(logs);
bitsi =  mantissai & (0x7ff&(accuracymask));
bitss =  mantissas & (0x7ff&(accuracymask));
infDone= (bitsi!=0) && (bitsi!=(0x7ff&(accuracymask)));
supDone= (bitss!=0) && (bitss!=(0x7ff&(accuracymask)));

/* Only one test, expected true */
if(__builtin_expect(infDone && supDone, TRUE))
  return result;

/* otherwise lauch accurate computation */
```

Avoiding a squaring of the number of tests

In each function, a few tests for special cases

(subnormals, infinities, etc.)

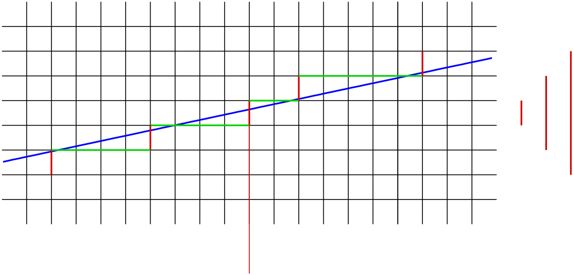
- Make common case fast:
  Fuse all the tests for both endpoints into one, with static prediction.
- If this branch is taken, simply call the existing CRLibm functions that round up and down.

Compared Itanium 2 timings

Library	exp	interval exp	log	interval log
`libm`	42	n/a	31	n/a
`fi_lib`	586	1038	619	1158
`CRlibm`	60	**69**	66	**96**

An exp/log iteration

$$\begin{cases} y_{n+1} = \log(x_n) \\ x_{n+1} = \exp(y_{n+1}) \end{cases}$$



Library	Interval bloat in ulps
`fi_lib`	13
`CRlibm`, RU and RD functions	2
`CRlibm` merged interval function	2

Conclusion and future work

Conclusions

- Itanium is nice for interval arithmetic
  - Nice floating-point unit for basic operations
  - Many many registers for parallel evaluation of elementary functions.
  - Predication, etc, for fused tests.
- `CRLibm` is nice for writing perfect interval functions
  - Safety: We are very happy to recycle our proofs
  - Tightness thanks to correct rounding
  - Performance-oriented algorithms

Future work

- Argue a lot: Is it worth writing tight functions ?
  - Yes in `CRLibm` (why not ? No additional work)
  - Not sure if you start from scratch
- A few implementation questions
  - The basic structure for FP intervals
  - Trigonometric argument reduction
  - Portability on a register-starved Pentium
- Publish `CRLibm0.14beta1`

Questions ?