An Ordinary Differential Equation Defined by a Computable Function whose Maximal Interval of Existence is Non-Computable

D. S. Graça[1,2] N. Zhong[3] J. Buescu[4]

[1]DM/FCT, University of Algarve, Portugal
[2]SQIG - Institute of Telecommunications, Portugal
[3]DMS, University of Cincinnati, U.S.A.
[4]CAMGSD, DM/IST, Technical University of Lisbon, Portugal

July 10, 2006

Motivation

- In general, given an Initial-Value Problem (IVP) defined with an ordinary differential equation (ODE)

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

  it is not possible to solve it explicitly
- Therefore, in applications, this kind of IVP is solved with *numerical methods*
- Schematically, these methods work out in the following manner: Given a time interval $[t_0, t_1]$ where the solution of the IVP is defined, compute this solution in that interval with an accuracy better than some $\varepsilon > 0$

Nevertheless, the task of determining $[t_0, t_1]$ is done in a very ad hoc way. For instance:

- For IVPs obtained as idealizations of physical systems, it is usually possible to get information about $[t_0, t_1]$ through the use of physical considerations
- In other situations, the system is solved numerically, and we then try to extract information about $[t_0, t_1]$ seeing when the solution starts diverging.

Obviously, this situation is not satisfactory!

Before specifying our problem, we need the notion of maximal interval

The maximal interval

Consider the IVP defined by the following ODE

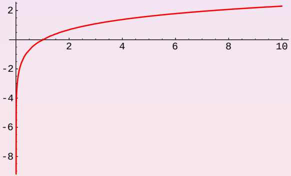
$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

Theorem (Maximal interval of existence)

Let E be an open subset of $\mathbb{R}^{m+1}$ and assume that $f : E \to \mathbb{R}^m$ is continuous on E and locally Lipschitz in the second argument. Then for each $(t_0, x_0) \in E$, the previous IVP has a unique solution $x(t)$ defined on a maximal interval (α, β), on which it is C^1. The maximal interval is open and has the property that, if $\beta < +\infty$ (resp. $\alpha > -\infty$), either $(t, x(t))$ approaches the boundary of E or $x(t)$ is unbounded as $t \to \beta^-$ (resp. $t \to \alpha^+$).

Our problem

Problem. If f and (t_0, x_0) are computable, does the previous IVP have a computable maximal interval? And will the respective solution be computable in that interval?



Graph of the solution

of IVP $x' = \frac{1}{x}$, where $x(1) = 0$. The maximal interval is $(0, +\infty)$ [the solution is $\log$]

Numerical Analysis

Is it the case that existing methods from Numerical Analysis already solve our problem?

No, because:

- The solutions are computed for a *compact interval* $[t_0, t_1]$ (due to the need of some global Lipschitz constant)
- The interval $[t_0, t_1]$ is given *a priori*

Main (negative) results

We show the following negative results:

Theorem

There is a function $f : \mathbb{R} \to \mathbb{R}$, computable and analytic, such that the solution of the IVP

$$x' = f(x), \; x(0) = 0$$

is defined in a non-computable maximal interval.

Theorem

Given an IVP of the previous type, with maximal interval (α, β), where f is analytic, f and (t_0, x_0) are computable, there is no algorithmic way to determine whether (α, β) is bounded or not.

Main (positive) results

On the other side, we prove a result that establish the degree of "non-computability" of the maximal interval of a IVP (Notation: r.e. = recursively enumerable)

Theorem

Let $E \subseteq \mathbb{R}^{m+1}$ be a r.e. open set and $f : E \to \mathbb{R}^m$ be a computable function that is also effectively locally Lipschitz in the second argument. Let (α, β) be the maximal interval of existence of the solution $x(t)$ of the previous IVP, where (t_0, x_0) is a computable point in E. Then (α, β) is a r.e. open interval and x is a computable function on (α, β).

The previous results, that say that (α, β) is a r.e. interval not necessarily computable (in the given conditions), can be interpreted in the following way:

1. It is possible to compute a succession $\{b_n\}_{n \in \mathbb{R}}$ with an algorithm, from f and (t_0, x_0), that converges to β (positive part)
2. Nevertheless there is no algorithm that, given b_n as input, outputs the distance from b_n to β (negative part)

To prove our results, besides techniques from ODEs, we use the Church-Turing thesis:

Everything that can be computed with an algorithm can be computed by a Turing machine (TM), and vice versa

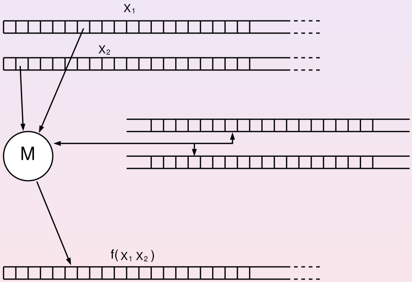
that is accepted as an axiom by the Theoretical Computer Science community (only in that way we are able to formalize the notion of algorithm). Once we accept this axiom, we are able to use results from the Theory of Computation. Namely, we use the fact that

There are problems (e.g. the Halting Problem, Hilbert's 10th Problem) which cannot be solved with the use of TMs

to arrive to the negative results. In the remaining of the talk, we provide technical details about the results.

Computable analysis: Type 2 machines

Let $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ be a representation of the rational numbers defined by: $\nu_{\mathbb{Q}}(\langle p, q, r \rangle) \mapsto (-1)^p \frac{q}{r+1}$



A tape represents a real number:

For a sequence (x_n) of integers, we write $(x_n) \rightsquigarrow x$ iff $\forall i, |x - \nu_{\mathbb{Q}}(x_i)| < \frac{1}{2^i}$

M behaves like a Turing Machine

Read-only one-way input tapes
Write-only one-way output tape.

Some definitions:

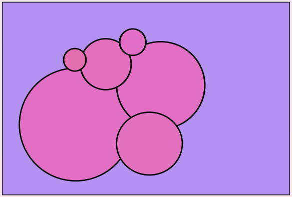
- A succession $\{r_n\}$ of rational numbers is called a ρ-name for a real number x if there are 3 functions a, b, c from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $r_n = (-1)^{a(n)} \frac{b(n)}{c(n)+1}$ and

$$|r_n - x| \leq \frac{1}{2^n}$$

- A real number x is called computable if a, b, c are computable functions.
- A succession $\{x_k\}_{k \in \mathbb{N}}$ of real numbers is called computable if there are three computable functions a, b, c from $\mathbb{N}^2$ to $\mathbb{N}$ such that, for all $k, n \in \mathbb{N}$,

$$\left| (-1)^{a(k,n)} \frac{b(k,n)}{c(k,n)+1} - x_k \right| \leq \frac{1}{2^n}.$$

- An open set $E \subseteq \mathbb{R}^m$ is called recursively enumerable (r.e.) if there are computable successions $\{a_n\}$ and $\{r_n\}$, $a_n \in E$ and $r_n \in \mathbb{Q}$ such that
$$E = \cup_{n=0}^{\infty} B(a_n, r_n)$$
and for all $n \in \mathbb{N}$, $\overline{B(a_n, r_n)} \subseteq E$



- A continuous function $f : E \to \mathbb{R}^m$ is computable if there is a Type-2 machine that translates each ρ-name of $x \in E$ to a ρ-name of $f(x)$
- Let $E = \cup B(a_n, r_n)$ be an open r.e. set. A function $f : E \to \mathbb{R}^m$ is called effectively locally Lipschitz in the second argument if there is a computable succession of positive integers $\{K_n\}$ such that

$$|f(t,x) - f(t,y)| \leq K_n |y - x|$$

for all $(t,x), (t,y) \in \overline{B(a_n, r_n)}$

Proof of the positive result

We want to show that the maximal interval is r.e., in the assumptions set in the theorem

- Since $\{a_n\}$ and $\{r_n\}$ are computable sequences and f is a computable function on E, both sequences $\{M_n\}$, $M_n = \max_{z\in\overline{B(a_n,r_n)}} |f(z)| + 1$, and $\{K_n\}$ are computable
- Then, using the classical proof of the existence of the solution of a given ODE, one can use the following algorithm to compute the maximal interval

1. Set $n = 0$
2. Compute an index l_n such that $x_n \in B(a_{l_n}, r_{l_n})$
3. Compute a time interval $[t_n, t_{n+1}]$ where the solution of $\dot{x} = f(t,x)$, $x(t_n) = x_n$ is defined
4. Set $x_{n+1} = x(t_{n+1})$ and increment n
5. Go to step 2

Notice that the time interval $[t_n, t_{n+1}]$ referred to in step 3 can be obtained from the proof of the existence of the solution of a given ODE. For instance, one can take $t_{n+1} = t_n + 2^{-K_{l_n}}/M_{l_n}$.

It is possible to show, by a contradiction argument, that the maximal interval is given by $(t_0, \beta) = \cup_{n=0}^{\infty}(t_0, t_n)$, i.e. that $t_n \to \beta$ as $n \to \infty$. The idea is the following:

- If t_n does not converge to β, it must converge to some $\gamma < \beta$
- Then, for some index $j \in \mathbb{N}$, one has $(\gamma, x(\gamma)) \in B(a_j, r_j)$. Since t_n converges increasingly to γ and it is known classically that $x : [t_0, \beta) \to E$ is continuous, one can find a sufficiently large n_0 such that $(t_{n_0}, x(t_{n_0})) \in B(a_j, r_j)$ and $t_{n_0} + 2^{-K_j}/M_j > \gamma$
- But, by construction, $t_{n_0+1} = t_{n_0} + 2^{-K_j}/M_j$. Thus $t_{n_0+1} > \gamma$. We have a contradiction

The solution x is computable because, given $t \in (t_0, \beta)$, we can:

- get an index $n \in \mathbb{N}$ such that $t \leq t_n$
- compute numerically the solution over $[t_0, t_n]$ to get the value of $x(t)$

Proof of the 1st negative result

Theorem

There is a function $f : \mathbb{R} \to \mathbb{R}$, computable and analytic, such that the solution of the IVP

$$x' = f(x), \ x(0) = 0$$

is defined in a non-computable maximal interval.

To simplify things, we prove the result for a function f that is computable, continuous, and effectively Lipschitz.

The proof of the analytic case can be found in a paper by the same authors, to appear in the "Transactions of the American Mathematical Society"

The proof of the 1st negative result relies on the use of the following lemma.

Lemma

Let $a : \mathbb{N} \to \mathbb{N}$ *be a computable function. Then there exists a computable function* $f : \mathbb{R} \to \mathbb{R}$ *such that the unique solution of the problem*

$$\dot{x} = f(x), \; x(0) = 0$$

is defined on a maximal interval $(-\alpha, \alpha)$ *with*

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{a(i)}}.$$

The 1st negative result follows from the lemma in the following manner.

- Pour-El and Richards showed that if $a : \mathbb{N} \to \mathbb{N}$ is a one to one recursive function generating a recursively enumerable nonrecursive set A, then $\alpha = \sum_{i=0}^{\infty} 2^{-a(i)}$ is a non computable real number. Since the previous Lemma shows how to obtain an IVP with maximal interval $(-\alpha, \alpha)$, this shows the first theorem

Proof of the Lemma

The idea is as follows: f is constructed piecewisely on intervals of the form $[i, i+1]$, $i \in \mathbb{N}$ (for negative values, we take $f(x) = f(|x|)$) such that the solution of the initial value problem

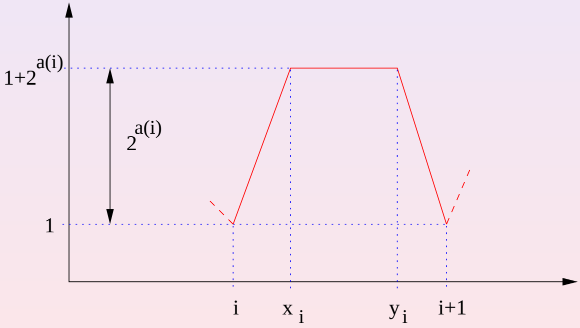
$$\dot{x} = f(x), \ \ x(0) = i$$

satisfies $x(2^{-a(i)}) = i + 1$, which implies that the solution of the problem $\dot{x} = f(x)$ and $x(0) = 0$ would satisfy $x(2^{-a(0)}) = 1$, $x(2^{-a(0)} + 2^{-a(1)}) = 2$, ..., or more generally

$$x\left(\sum_{i=0}^{n} 2^{-a(i)}\right) = n + 1, \quad \text{for all } n \in \mathbb{N}.$$

(Notice that f does not depend on t and therefore the solution is invariant under time translations.) If we take $\alpha = \sum_{i=0}^{\infty} 2^{-a(i)}$, then $x(t) \to \infty$ as $t \to \alpha^-$ and therefore the maximal interval must be $(-\alpha, \alpha)$.

We now construct the desired function f on intervals of the form $[i, i+1]$, $i \in \mathbb{N}$. As we recall that a computable function must be continuous, we need to glue the values of f at the endpoints of these intervals. This is achieved by assuming that $f(i) = 1$ for $i \in \mathbb{N}$



Hence, f can be defined by

$$f(x) = \begin{cases} 1 + 2^{a(i)} \frac{x-i}{x_i - i} & \text{if } x \in [i, x_i) \\ 1 + 2^{a(i)} & \text{if } x \in [x_i, y_i), \\ 1 + 2^{a(i)} - 2^{a(i)} \frac{x - y_i}{i+1-y_i} & \text{if } x \in [y_i, i+1), \end{cases}$$

where

$$x_i = i + \frac{1 - \Delta_i}{2}, \qquad y_i = i + \frac{1 + \Delta_i}{2},$$

and

$$0 < \Delta_i = \frac{1 - \ln(2^{a(i)} + 1)}{1 - (1 + 2^{a(i)})^{-1} - \ln(1 + 2^{a(i)})} < 1.$$

(Here we supposed $a(i) \geq 1$. If $a(i) \geq 0$, then consider the function $a' : \mathbb{N} \to \mathbb{N}$ defined by $a'(i) = a(i) + 1$ and use a variable substitution $\tilde{t} = t/2$ on the corresponding ODE)

Proof of the 2nd negative result

- Consider the IVP $x' = f(t, x)$, $x(t_0) = x_0$
- Suppose that there is a TM that, on input $\langle f, t_0, x_0 \rangle$, outputs 1 if $\beta < \infty$ and 0 otherwise
- Consider the following undecidable problem: "Let $\psi : \mathbb{N}^2 \to \mathbb{N}$ be the function generated by an Universal TM. Then, given $i \in \mathbb{N}$, decide if $\psi(i, i)$ is defined". Let M_1 be a TM computing ψ
- Let $g : \mathbb{N}^2 \to \mathbb{N}$ be the recursive function defined by

$$g(i,j) = \begin{cases} 0 & \text{if } M_1 \text{ stops on input } (i,i) \text{ in } \leq j \text{ steps} \\ 1 & \text{otherwise} \end{cases}$$

Then

$$\psi(i,i) \text{ is defined} \quad \text{iff} \quad \exists j_0 \in \mathbb{N} \quad (\forall j \geq j_0,\ g(i,j) = 0)$$

- Consider the computable succession of analytic functions $\{\varphi_i\}$, where $\varphi_i : \mathbb{R} \to \mathbb{R}$ is defined by

$$\varphi_i(x) = \sum_{n=1}^{\infty} a_{i,2n}\, x^{2n}, \quad \text{where } a_{i,n} = \left(\frac{1}{3}\right)^{n^2} + \left(\frac{1}{3}\right)^{n} g(i,n).$$

- Note that the convergence radius of φ_i is $+\infty$ iff $\psi(i,i)$ is defined
- Then, using the hypothesis, there is a TM M_2 that, on input i, outputs 1 if the maximal interval of the IVP $x' = \varphi_i'(t)$, $x(0) = 0$ is limited and 0 otherwise
- But then, since

$$M_2 \text{ with input } i \text{ outputs } \begin{cases} 0 & \text{if } \psi(i,i) \text{ is defined} \\ 1 & \text{otherwise} \end{cases}$$

M_2 decides an undecidable problem, which is absurd.

Conclusion

- We constructed a computable counterpart of the fundamental existence-uniqueness theory for ODEs
- We showed that ODEs defined with computable analytic functions can have non-computable properties
- It would be interesting to have results on the *complexity* level
- It would be interesting to know to which subclass of the analytic function the previous properties become computable

Thank you!